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LINEAR STABILIZATION OF PROGRAMMED MOTIONS OF NON-LINEAR CONTROLLED DYNAMICAL SYSTEMS†

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A non-linear controlled dynamical system (NCDS), describing the dynamics of a wide class of non-linear mechanical and electromechanical systems, is considered. A technique is proposed for synthesizing control laws with linear feedback according to state, which describe the stabilization of programmed motions of such systems. A non-singular linear transformation of the state space is constructed, bringing the initial NCDS in deviations (from its programmed motion and programmed control) to a certain NCDS of a special form, which is convenient for analysing and synthesizing control laws governing the motion of the system. A NCDS of canonical form is separated out from the initial NCDS in deviations. The aforementioned non-singular linear transformation of coordinates of the state space and the method of Lyapunov functions are used to synthesize control laws with linear state feedback, which guarantee global symptotic stability of an equilibrium position of a NCDS of canonical form and asymptotic stability in the large of a NCDS of special form and of the initial NCDS in deviations. Estimates are given for the domain of asymptotic stability in the large of the equilibrium positions of a NCDS of special form, of the initial NCDS in deviations, and of programmed motions of the initial NCDS, closed by the synthesized stabilizing controls.

1. FORMULATION OF THE PROBLEM

The dynamics of a wide class of mechanical and electromechanical systems is described by a system of non-linear ordinary differential equations in Cauchy form, of the type

$$\dot{z} = F(z, u, t), \quad z(t_0) = z_0, \quad t \geq t_0 \geq 0 \tag{1.1}$$

where z_0, $z = z(t)$ are the n-dimensional state vectors of the system at the initial and current instants of time, u is an n-dimensional vector of controls and F is an n-dimensional vector-valued function which, for an admissible control, satisfies existence and uniqueness conditions for the solvability of system (1.1) and describes the properties of the controlled object.

Suppose we are given (or have constructed) a programmed motion (PM)

$$z_p = z_p(t), \quad t \geq t_0 \tag{1.2}$$

which is a particular solution of system (1.1) for some admissible programmed control

$$u_p = u_p(t), \quad t \geq t_0 \tag{1.3}$$

and initial condition $z_{p0} = z_p(t_0)$. The PM $z_p(t)$ will be called the unperturbed motion, and any other motion $z(t)$ of system (1.1) governed by admissible controls will be called a perturbed (real) motion.

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The quantities

$$e = z - z_p, \quad e_u = u - u_p \tag{1.4}$$

and perturbations, that is, deviations of the real (perturbed) motion z and the control u from their programmed values. They satisfy the following ordinary differential equations in deviations

$$\dot{e} = F_e(e, e_u, t), \quad e(t_0) = e_0, \quad t \geq t_0 \tag{1.5}$$

where

$$F_e(e, e_u, t) = F(e + z_p, e_u + u_p, t) - F(z_p, u_p, t) \tag{1.6}$$

and $F_e(0, 0, t) \equiv 0$. It follows from formula (1.6) that under the control $e_u = 0$ system (1.5), (1.6) has the motion $e \equiv 0$.

The transformations (1.4) reduce the problem of investigating the motions $z(t)$ of a non-linear controlled dynamical system (NCDS) (1.1) in the neighbourhood of any selected PM $z_p(t)$ to the problem of investigating the solutions $e = e(t)$ of a NCDS in deviations (1.5), (1.6) in the neighbourhood of its equilibrium position $e = 0$. In what follows, therefore, the main restrictions and propositions will be formulated for NCDS in deviations (1.5), (1.6).

For a wide class of mechanical and electromechanical systems (such as electromechanical manipulator robots; see Appendix, Section 5), the structure of the equations of the NCDS in deviations (1.5), (1.6) is such that

$$e = \mathrm{col}(e_1, \ldots, e_r), \quad n = mr \tag{1.7}$$

$$F_e(e, e_u, t) = \mathrm{col}(F_{e1}(e^2, t), \ldots, F_{e, r-1}(e^r, t), F_{er}(e^r, e_u, t)) \tag{1.8}$$

$$\begin{aligned} &F_{e1}(e^2, t) = g_{e1}(e^1, t) + \bar{P}_{012}(t)e_2 \\ &F_{ek}(e^{k+1}, t) = g_{ek}(e^k, t) + \bar{P}_{0k,k+1}(e^{k-1}, t)e_{k+1}, \quad k = 2, \ldots, r-1 \\ &F_{er}(e^r, e_u, t) = g_{er}(e^r, t) + \bar{P}_{0r,r+1}(e^{r-1}, t)e_u \end{aligned} \tag{1.9}$$

in which $e_k = \mathrm{col}(e_{k1}, \ldots, e_{km})$ and $e^k = \mathrm{col}(e_1, \ldots, e_k)$ are m- and mk-dimensional column vectors, the m-dimensional vector-valued functions F_{ek} $(k = 1, \ldots, r)$ (1.9) are continuous and sufficiently many times continuously differentiable with respect to their arguments, and the $m \times m$ matrix-valued functions $\bar{P}_{0k,k+1}$ $(k = 1, \ldots, r)$ may be represented in the form

$$\bar{P}_{012}(t) = A_1(t)B_1; \quad \bar{P}_{0k,k+1}(e^{k-1}, t) = A_k(e^{k-1}, t)B_k, \quad k = 2, \ldots, r \tag{1.10}$$

where

$$\begin{aligned} &A_1(t) = A_1^*(t) > 0, \quad t \geq t_0 \\ &A_k(e^{k-1}, t) = A_k^*(e^{k-1}, t) > 0, \quad \forall e^{k-1} \in R^{m(k-1)}, \quad t \geq t_0, \quad k = 2, \ldots, r \end{aligned} \tag{1.11}$$

A_k $(k = 1, \ldots, r)$ are symmetric positive-definite $m \times m$ matrix-valued functions such that

$$\begin{aligned} &|A_1(t)| \leq k_{A1}, \quad t \geq t_0 \\ &|A_k(e^{k-1}, t)| \leq k_{Ak}, \quad \forall e^{k-1} \in R^{m(k-1)}, \quad t \geq t_0; \quad k = 2, \ldots, r \end{aligned} \tag{1.12}$$

$0 < k_{Ak} < \infty$ $(k = 1, \ldots, r)$ are certain constants; analogous estimates hold for the partial derivatives of their elements – scalar functions a_{kij} $(k = 1, \ldots, r; i, j = 1, \ldots, m)$ – with respect to their arguments; the asterisk denotes transposition and B_k $(k = 1, \ldots, r)$ are non-singular constant $m \times m$ matrices, that is,

$$\mathrm{rank} B_k = m, \quad k = 1, \ldots, r \tag{1.13}$$

$R^{m(k-1)}$ is real Euclidean $m(k-1)$-space; throughout, $|A| = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$ and $|a| = (a_1^2 + \ldots + a_n^2)^{1/2}$ are the moduli (Euclidean norms) of a real matrix $A = \|a_{ij}\|_{i=1,\ldots,n;\, j=1,\ldots,m}$ of order $n \times m$ and a real vector $a = \mathrm{col}(a_1, \ldots, a_n) \in R^n$.

Each of the m-vector-valued functions g_{ek} ($k = 1, \ldots, r$) satisfies the following estimates for all possible values of its arguments

$$|g_{ek}(e^k, t)| \le k_{gek1}|e^k| + k_{gek2}|e^k|^2, \quad \forall e^k \in R^{mk}, \quad t \ge t_0 \tag{1.14}$$

where k_{gekj} ($j = 1, 2$) are certain constants such that

$$0 \le k_{gekj} < \infty, \quad k = 1, \ldots, r; \quad 0 < k_{ge2} = \sum_{k=1}^{r} k_{gek2} < \infty \tag{1.15}$$

In what follows, the control law

$$u = u(z, t) = u_p(t) + \Gamma_0(z - z_p), \quad t \ge t_0 \tag{1.16}$$

where

$$\Gamma_0 = \|\Gamma_{01}, \ldots, \Gamma_{0r}\| \tag{1.17}$$

(Γ_0 is the constant $m \times n$ partitioned matrix of the gains of the feedback loops; Γ_{0k} ($k = 1, \ldots, r$) are the $m \times m$ blocks), for the initial NCDS (1.1), (1.6)–(1.15) and the control law

$$e_u = e_u(e) = \Gamma_0 e \tag{1.18}$$

for the initial NCDS is deviations (1.5)–(1.15) have the structure of linear feedback control laws depending on the states z and e, respectively.

We shall say that a PM $z_p(t)$ (1.2) of system (1.1), (1.6)–(1.15) is stabilizable by a control law u (1.15)–(1.17) with linear feedback depending on the state vector $z(t)$ (or, respectively, that the equilibrium position $e = 0$ of system (1.5)–(1.15) is stabilizable by a control law e_u (1.18), (1.17) with linear feedback depending on the state vector $e(t)$, if the control law guarantees asymptotic stability in the large of the PM $z_p(t)$ of system (1.1), (1.6)–(1.15) (or, respectively, of the equilibrium position $e = 0$) of system (1.5)–(1.15)) according to Definition 5 presented below in Section 3.

In what follows, we shall formulate criteria for the equilibrium position $e = 0$ of the initial NCDS in deviations (1.15)–(1.15) to be stabilizable by a control law e_u (1.18), (1.17) with linear feedback depending on the state e (resp., for a PM z_p (1.2) of the initial NCDS (1.1), (1.6)–(1.15) to be stabilizable by a control law u (1.16), (1.17) with linear feedback depending on the state z). Estimates will be given for the domain of asymptotic stability in the large of the equilibrium position $e = 0$ of the closed NCDS in deviations (1.5)–(1.15), (1.18), (1.17) (resp., of the PM z_p (1.2) of the closed-loop NCDS (1.1), (1.6)–(1.17)).

2. REDUCTION OF THE INITIAL NCDS IN DEVIATIONS TO AN NCDS OF SPECIAL FORM

For a further consideration of the NCDS in deviations (1.5)–(1.15), we will write it in the form of the system

$$\dot{e} = \bar{P}_0(e^{r-2}, t)e + \bar{Q}_0(e^{r-1}, t)e_u + g_e(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{2.1}$$

where

$$\bar{P}_0(e^{r-2}, t)e + \bar{Q}_0(e^{r-1}, t)e_u + g_e(e, t) \equiv F_e(e, e_u, t) \tag{2.2}$$

F_e is the vector-valued function (1.6)–(1.15);

$$\bar{P}_0(e^{r-2},t) = \left\| \begin{matrix} O & \bar{P}_{012}(t) & O & \dots & \dots & O \\ O & O & \bar{P}_{023}(e^1,t) & O & \dots & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ O & O & \dots & \dots & O & \bar{P}_{0,r-1,r}(e^{r-2},t) \\ O & O & \dots & \dots & O & O \end{matrix} \right\| \tag{2.3}$$

$$\bar{Q}_0(e^{r-1},t) = \left\| \begin{matrix} O \\ \bar{P}_{0r,r+1}(e^{r-1},t) \end{matrix} \right\| \tag{2.4}$$

are partitioned matrix functions of orders $n \times n$ and $n \times m$, respectively, $\bar{P}_{0k,\,k+1}$ $(k = 1, \dots, r)$ are $m \times m$ blocks of the form (1.10)–(1.13), O is the zero matrix of appropriate dimensionality, and

$$g_e(e,t) = \mathrm{col}(g_{e1}(e^1,t), g_{e2}(e^2,t), \dots, g_{er}(e^r,t)) \tag{2.5}$$

is an n-vector-valued function for which, thanks to relations (1.14) and (1.15), we have the estimate

$$|g_e(e,t)| \le \sum_{k=1}^{r} |g_{ek}(e^k,t)| \le k_{ge1}|e| + k_{ge2}|e|^2, \quad \forall e \in R^n, \quad t \ge t_0 \tag{2.6}$$

where

$$k_{gej} = \sum_{k=1}^{r} k_{gekj}, \quad j = 1,2; \quad 0 \le k_{ge1} < \infty, \quad 0 < k_{ge2} < \infty \tag{2.7}$$

We apply to system (2.1)–(2.7) a non-singular linear transformation of coordinates of the state space, of the form

$$e_x = Se \ (e = S^{-1}e_x = Re_x) \tag{2.8}$$

where

$$e_x = \mathrm{col}(e_{x1}, \dots, e_{xr}) \tag{2.9}$$

$e_{xk} = \mathrm{col}(e_{xk1}, \dots, e_{xkm})$ are n- and m- dimensional vectors, S and R are non-singular constant partitioned-triangular $n \times n$ matrices of the form

$$S = \left\| \begin{matrix} I_m & O & \dots & & \dots & O \\ S_{21} & I_m & O & \dots & \dots & O \\ S_{32}S_{21} & S_{32} & I_m & O & \dots & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ S_{r-1,r-2}S_{r-2,r-3}\dots S_{21} & S_{r-1,r-2}S_{r-2,r-3}\dots S_{32} & \dots & S_{r-1,r-2} & I_m & O \\ S_{r,r-1}S_{r-1,r-2}\dots S_{21} & S_{r,r-1}S_{r-1,r-2}\dots S_{32} & \dots & \dots & S_{r,r-1} & I_m \end{matrix} \right\| = \\ = \|S_{kl}\|_{k,l=1,\dots,r} \tag{2.10}$$

where

$$\begin{aligned} &S_{kl} = O, \quad k = 1, \ldots, r-1; \quad l = k+1, \ldots, r \\ &S_{kk} = I_m, \quad k = 1, \ldots, r; \quad S_{kl} = S_{k,k-1}S_{k-1,l} = S_{k,k-1}S_{k-1,k-2}\ldots S_{l+1,l} \\ &k = 3, \ldots, r; \quad l = 1, \ldots, k-2 \end{aligned} \tag{2.11}$$

$S_{k+1,k}$ ($k = 1, \ldots, r-1$) are $m \times m$ blocks of the form indicated below in Lemma 1;

$$R = S^{-1} = \begin{Vmatrix} I_m & O & \ldots & & \ldots & O \\ -S_{21} & I_m & O & \ldots & \ldots & O \\ O & -S_{32} & I_m & O & \ldots & O \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ O & \ldots & O & -S_{r-1,r-2} & I_m & O \\ O & \ldots & \ldots & O & -S_{r,r-1} & I_m \end{Vmatrix} \tag{2.12}$$

and I_m the $m \times m$ identity matrix.

Then the initial NCDS is deviations (2.1)–(2.7) is transformed to a NCDS of special form

$$\dot{e}_x = P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{2.13}$$

where

$$P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u + g_{ex}(e_x, t) \equiv F_{ex}(e_x, e_u, t) = SF_e(Re_x, e_u, t) \tag{2.14}$$

F_e is the vector-valued function (1.6)–(1.15); and

$$P_1(e_x^{r-2}, t) = S\bar{P}_0(\sigma^{r-2}(e_x^{r-2}), t)R =$$

$$= \begin{Vmatrix} P_{111}(t) & P_{112}(t) & O & \ldots & \ldots & O \\ P_{121}(t) & P_{122}(e_x^1, t) & P_{123}(e_x^1, t) & O & \ldots & O \\ \vdots & \vdots & P_{133}(e_x^2, t) & \ddots & \ddots & \vdots \\ & & \vdots\ddots & & \ddots & \vdots \\ & & & \ddots & P_{1,r-2,r-1}(e_x^{r-3}, t) & O \\ \vdots & \vdots & \vdots & & P_{1,r-1,r-1}(e_x^{r-2}, t) & P_{1,r-1,r}(e_x^{r-2}, t) \\ P_{1r1}(t) & P_{1r2}(e_x^1, t) & P_{1r3}(e_x^2, t) & \ldots & P_{1,r,r-1}(e_x^{r-2}, t) & P_{1rr}(e_x^{r-2}, t) \end{Vmatrix} =$$

$$= \|P_{1kl}\|_{k,l=1,\ldots,r} \tag{2.15}$$

is a partitioned $n \times n$ matrix-valued function whose $m \times m$ blocks have the form

$$\begin{aligned} &P_{111} \equiv P_{111}(t) = -\bar{P}_{012}(t)S_{21}, \quad P_{112} \equiv P_{112}(t) = \bar{P}_{012}(t) \\ &P_{1k1} \equiv P_{1k1}(t) = -S_{k1}\bar{P}_{012}(t)S_{21}, \quad k = 2, \ldots, r \\ &P_{1,k,k+1} \equiv P_{1k,k+1}(e_x^{k-1}, t) = \bar{P}_{0,k,k+1}(\sigma^{k-1}(e_x^{k-1}), t), \quad k = 2, \ldots, r-1 \\ &P_{1kk} \equiv P_{1kk}(t)(e_x^{k-1}, t) = S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}), t) - \\ &- \bar{P}_{0,k,k+1}(\sigma^{k-1}(e_x^{k-1}), t)S_{k+1,k}, \quad k = 2, \ldots, r-1 \end{aligned} \tag{2.16}$$

$$P_{1kl} = O, \quad k = 1, \ldots, r-2; \quad l = k+2, \ldots, r$$

$$P_{1kl} \equiv P_{1kl}(e_x^{l-1}, t) = S_{k,l-1}\bar{P}_{0,l-1,l}(\sigma^{l-2}(e_x^{l-2}), t) -$$
$$- S_{kl}\bar{P}_{0l,l+1}(\sigma^{l-1}(e_x^{l-1}), t)S_{l+1,l}, \quad k = 3, \ldots, r; \quad l = 2, \ldots, k-1$$

$$P_{1rr} \equiv P_{1rr}(e_x^{r-2}, t) = S_{r,r-1}\bar{P}_{0,r-1,r}(\sigma^{r-2}(e_x^{r-2}), t)$$

$e_x^k = \mathrm{col}(e_{x1}, \ldots, e_{xk})$; throughout,

$$\sigma^k \equiv \sigma^k(e_x^k) = \mathrm{col}(\sigma_1(e_{x1}), \sigma_2(e_{x1}, e_{x2}), \ldots, \sigma_k(e_{x,k-1}, e_{xk})) =$$
$$= H_k R e_x = H_k e = e^k = \mathrm{col}(e_1, \ldots, e_k) \tag{2.17}$$

and ($\sigma_1(e_{x1}) = e_{x1} = e_1$, $\sigma_k(e_{x,k-1}, e_{xk}) = -S_{k,k-1}e_{x,k-1} + I_m e_{xk} = e_k$, $k = 2, \ldots, r$) is an mk-vector-valued function, where $H_k = ||I_{km}, O||$ is a constant $(km) \times n$ partitioned matrix; everywhere,

$$\bar{P}_{012}(\sigma^0(e_x^0), t) \equiv \bar{P}_{012}(t) \tag{2.18}$$

is an $m \times m$ block; when relations (2.4), (2.10) and (2.11) are taken into account, Q_1 is a partitioned matrix of the form

$$Q_1(e_x^{r-1}, t) = S\bar{Q}_0(\sigma^{r-1}(e_x^{r-1}), t) = \bar{Q}_0(\sigma^{r-1}(e_x^{r-1}), t) \tag{2.19}$$

$$g_{ex}(e_x, t) = Sg_e(Re_x, t) \tag{2.20}$$

is an n-vector-valued function which, when relations (2.5)–(2.7) and (2.8)–(2.12) are taken into account, satisfies the estimate

$$|g_{ex}(e_x, t)| = |Sg_e(Re_x, t)| \le |S||g_e(Re_x, t)| \le |S|(k_{ge1}|Re_x| + k_{ge2}|Re_x|^2) \le$$
$$\le k_{gex1}|e_x| + k_{gex2}|e_x|^2, \quad \forall e_x \in R^n, \quad t \ge t_0 \tag{2.21}$$

where k_{gexj} ($j = 1, 2$) are certain constants such that

$$0 \le k_{gex1} = |S||R|k_{ge1} < \infty, \quad 0 < k_{ge2} = |S||R|^2 k_{ge2} < \infty \tag{2.22}$$

3. DEFINITIONS AND AUXILIARY LEMMA ON THE SYMPTOTIC STABILITY IN THE LARGE OF THE EQUILIBRIUM POSITION OF A NON-LINEAR DYNAMICAL SYSTEM

Let us consider a non-linear dynamical system

$$\dot{e} = f(e, t) + g(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{3.1}$$

where e_0, $e = e(t) \equiv e(t; e_0, t_0)$ are the n-dimensional state vectors of the system at the initial and current instants of time and f and g are n-vector-valued functions with

$$f(0, t) = g(0, t) \equiv 0, \quad |g(e, t)| \le k_{g1}|e| + k_{g2}|e|^2, \quad \forall e \in R^n, \quad \forall t \ge t_0$$
$$0 \le k_{g1} < \infty, \quad 0 < k_{g2} < \infty \tag{3.2}$$

where k_{gj} are certain constants.

It is assumed that a solution of the Cauchy problem for system (3.1), (3.2) exists and is unique.

We will give some definitions [1–6] that will be used below to investigate the behaviour of the solution of system (3.1), (3.2).

Definition 1 [1; 2, p. 9; 3, pp. 66, 67]. The equilibrium position $e = 0$ of system (3.1), (3.2) is said to be Lyapunov-stable if, for any number $\varepsilon > 0$ and any $t_0 \geq 0$, a number $\delta \equiv \delta(\varepsilon, t_0) > 0$ exists such that if

$$|e_0| < \delta \equiv \delta(\varepsilon, t_0)$$

then

$$|e(t; e_0, t_0)| < \varepsilon \quad \text{for all} \quad t \geq t_0$$

Otherwise, the equilibrium position $e = 0$ of system (3.1), (3.2) is unstable.

Definition 2 [1; 3, p. 68]. The equilibrium position $e = 0$ of system (3.1), (3.2) is said to be asymptotically stable as $t \to +\infty$ if it is Lyapunov-stable and, for any $t_0 \geq 0$, a positive number $\Delta \equiv \Delta(t_0) \leq \delta(\varepsilon, t_0)$ exists such that, if

$$|e_0| < \Delta$$

then

$$|e(t; e_0, t_0)| \to 0 \quad \text{as} \quad t \to +\infty \tag{3.3}$$

Definition 3 [3, p. 68; 4, p. 69]. The domain

$$\Omega_a = \{e_0 \in R^n : |e_0| < \Delta \equiv \Delta(t_0)\}$$

(for fixed t_0) such that condition (3.3) is satisfied is called the domain of attraction or domain of asymptotic stability (DAS) of the equilibrium position $e = 0$ of system (3.1), (3.2).

Definition 4 [3, p. 68; 4, p. 69]. If the equilibrium position $e = 0$ of system (3.1), (3.2) is asymptotically stable as $t \to +\infty$ and all the solutions $e = e(t)$ $(0 \leq t_0 \leq t < \infty)$ have property (3.3), that is,

$$\Delta = \infty$$

then the equilibrium position $e = 0$ is said to be globally asymptotically stable, that is, system (3.1), (3.2) is said to be globally asymptotically stable if

$$\Omega_a = R^n$$

Definition 5 [5, p. 29]. Let $\rho_0 > 0$ be a given position number. The equilibrium position $e = 0$ of system (3.1), (3.2) is said to be asymptotically stable in the large if it is Lyapunov-stable and condition (3.3) holds for any initial conditions e_0 in the domain

$$\Omega_0 = \{e_0 \in R^n : |e_0| < \rho_0\}$$

Below, when studying the behaviour of solutions of various NCDSs, we will use the method of Lyapunov functions [1–6], which enables us not only to establish asymptotic stability, but also to obtain estimates of the DAS of the unperturbed motions of the systems (in particular, when solving problems of asymptotic stability in the large, when the domain of initial perturbations cannot be considered to be as small as desired). For example, according to the method [2, p. 21; 5, pp. 29, 30; 6, p. 149], if a real continuously differentiable scale function $\upsilon(e, t)$ is positive-definite [3, p. 235], $\upsilon(0, t) = 0$ and along a non-trivial solution $e(t)$ of system (3.1), (3.2) the function

$$\dot{\upsilon} = \dot{\upsilon}(e(t), t) = \frac{\partial \upsilon(e(t), t)}{\partial t} + \frac{\partial \upsilon(e(t), t)}{\partial e(t)}(f(e, t) + g(e, t)) = w(e(t), t) = w(e, t)$$

is negative-definite [3, p. 236] in a bounded domain

$$\Omega_0 = \{e \in R^n : \upsilon(e, t) < \rho_0, t \geq t_0\} \tag{3.4}$$

(where $\rho_0 > 0$ is a real number), then the equilibrium position $e = 0$ of system (3.1), (3.2) is asymptotically stable in the large and the domain Ω_0 is an estimate of its DAS or domain of attraction [2, p. 21]. This means that all trajectories $e(t; e_0, t_0)$ of system (3.1), (3.2) that begin at $t = t_0$ in the domain Ω_0 (3.4) will tend to the origin ($e = 0$) as $t \to \infty$, that is,

$$|e(t; e_0, t_0)| \to 0 \quad \text{as} \quad t \to \infty, \quad e(t_0) = e_0 \in \Omega_0$$

Auxiliary Lemma. Let us assume that a real continuously differentiable scalar function $\upsilon(e, t)$ and real numbers $\varepsilon_{\upsilon i} > 0$ ($i = 1, 2, 3$), $\alpha_0 > 0$, $0 < \nu_0 < 1$ exist such that

(1) $\varepsilon_{\upsilon 1}|e| \le \upsilon(e, t) \le \varepsilon_{\upsilon 2}|e|$, $e \in R^n$, $t \ge t_0$ $\upsilon(0, t) = 0$;

(2) $\left|\dfrac{\partial \upsilon(e, t)}{\partial e}\right| \le \varepsilon_{\upsilon 3}$, $\dfrac{\partial \upsilon(e, t)}{\partial e} \ne 0$, $|e| \ne 0$. $t \ge t_0$;

(3) in estimate (3.2) for the vector-valued function $g(e, t)$ the coefficients k_{gi} ($j = 1, 2$) satisfy the inequalities

$$0 \le k_{g1} < (1 - \nu_0)\alpha_0 \varepsilon_{\upsilon 1} \varepsilon_{\upsilon 3}^{-1}, \quad 0 < \nu_0 < 1; \quad 0 < k_{g2} < \infty$$

(4) the derivative with respect to time t of the function $\upsilon(e(t), t)$ along trajectories of the system

$$\dot{e} = f(e, t), \quad e(t_0) = e_0, \quad t \ge t_0$$

evaluated along a non-trivial solution $e(t) = e(t; e_0, t_0)$ of the system, satisfies the estimate

$$\frac{d}{dt}\upsilon(e(t), t) = \frac{\partial \upsilon(e(t), t)}{\partial t} + \frac{\partial \upsilon(e(t), t)}{\partial e(t)} f(e(t), t) \le -\alpha_0 \upsilon(e(t), t), \quad t \ge t_0$$

Then

(1) the equilibrium position $e = 0$ of system (3.1), (3.2) is asymptotically stable in the large;

(2) the DAS of the equilibrium position $e = 0$ of system (3.1), (3.2) is the set

$$\Omega_0 = \{e \in R^n\colon \upsilon(e, t) < \rho_0, \ t \ge t_0\}; \quad \rho_0 = \frac{\varepsilon_{\upsilon 1}}{k_{g2}}\left[(1 - \nu_0)\alpha_0 \frac{\varepsilon_{\upsilon 1}}{\varepsilon_{\upsilon 3}} - k_{g1}\right] > 0 \tag{3.5}$$

(3) a non-trivial solution $e(t)$ of system (3.1), (3.2) satisfies the estimate

$$|e(t)| \le \beta_0 e^{-\gamma_0(t - t_0)}|e(t_0)|, \quad e(t_0) = e_0 \in \Omega_0, \quad t \ge t_0; \quad \beta_0 = \varepsilon_{\upsilon 2}\varepsilon_{\upsilon 1}^{-1}, \quad \gamma_0 = \nu_0 \alpha_0$$

where Ω_0 is the set (3.5).

Proof. Taking conditions 1–4 of the lemma into account, let us evaluate the derivative with respect to t of the function $\upsilon \equiv \upsilon(e(t), t)$ along trajectories of system (3.1), (3.2). This gives

$$\begin{aligned}
\dot{\upsilon} &= \frac{\partial \upsilon}{\partial t} + \frac{\partial \upsilon}{\partial e}(f(e, t) + g(e, t)) \le -\alpha_0 \upsilon + \left|\frac{\partial \upsilon}{\partial e}\right| |g(e, t)| \le -\alpha_0 \upsilon + \\
&+ \varepsilon_{\upsilon 3}(k_{g1}|e| + k_{g2}|e|^2) \le -\nu_0 \alpha_0 \upsilon - (1 - \nu_0)\alpha_0 \upsilon + \varepsilon_{\upsilon 3}\varepsilon_{\upsilon 1}^{-1}\upsilon(k_{g1} + k_{g2}\varepsilon_{\upsilon 1}^{-1}\upsilon) \le \\
&\le -\nu_0 \alpha_0 \upsilon + \upsilon\{\varepsilon_{\upsilon 3}\varepsilon_{\upsilon 1}^{-2}k_{g2}\upsilon + [\varepsilon_{\upsilon 3}\varepsilon_{\upsilon 1}^{-1}k_{g1} - (1 - \nu_0)\alpha_0]\} \le -\nu_0 \alpha_0 \upsilon = \\
&= -\gamma_0 \upsilon, \quad e(t_0) = e_0 \in \Omega_0, \quad t \ge t_0; \quad 0 < \nu_0 < 1, \quad \gamma_0 = \nu_0 \alpha_0
\end{aligned} \tag{3.6}$$

where Ω_0 is the set (3.5).

Hence, by condition 1 of the lemma, we deduce the validity of parts 1 and 2.

The estimate (3.6) and condition 1 of the lemma imply the inequality

$$\upsilon(e, t) \le e^{-\gamma_0(t - t_0)}\upsilon(e_0, t_0), \quad e(t_0) = e_0 \in \Omega_0, \quad t \ge t_0 \tag{3.7}$$

where Ω_0 is the set (3.5). It follows from inequality (3.7) and the estimates for the function $\upsilon(e, t)$ in condition 1 of the lemma that the third part of the lemma also holds. The lemma is proved.

4. CRITERIA FOR THE STABILIZABILITY OF NCDS

1. We will first consider the behaviour of a solution $e_x(t)$ of the NCDS

$$\dot{e}_x = P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u, \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{4.1}$$

(where e_x is the state vector (2.8) of the system, P_1 and Q_1 are the matrix-valued functions (2.15)–(2.18) and (2.19)), closed by a control law e_u (1.18), (1.17) represented, with due note of (2.8)–(2.12), in the form

$$e_u = \Gamma_0 e = e_{ux} \equiv \Gamma_0 R e_x = \bar{\Gamma}_0 e_x \tag{4.2}$$

where Γ_0 is a constant $m \times n$ matrix consisting of $m \times m$ blocks:

$$\bar{\Gamma}_0 = \Gamma_0 R = \|\bar{\Gamma}_{01}, \ldots, \bar{\Gamma}_{0r}\| \tag{4.3}$$

$$\bar{\Gamma}_{0k} = \Gamma_{0k} - \Gamma_{0,k+1} S_{k+1,k}, \quad k = 1, \ldots, r-1; \quad \bar{\Gamma}_{0r} \tag{4.4}$$

and the equations of the transients (in the above closed-loop system) have the form

$$\dot{e}_x = P(e_x^{r-1}, t)e_x, \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{4.5}$$

Here

$$P(e_x^{r-1}, t) = P_1(e_x^{r-2}, t) + Q_1(e_x^{r-1}, t)\bar{\Gamma}_0 = P_1(e_x^{r-2}, t) + P_2(e_x^{r-1}, t) \tag{4.6}$$

is an $n \times n$ matrix-valued function, where P_1 is the matrix-valued function (2.15)–(2.18) and, when relations (2.19), (2.4) and (2.17) are taken into account

$$P_2(e_x^{r-1}, t) = Q_1(e_x^{r-1}, t)\bar{\Gamma}_0 = \bar{Q}_0(\sigma^{r-1}(e_x^{r-1}), t)\bar{\Gamma}_0 = \left\| \begin{matrix} O \\ \bar{P}_{0r,r+1}(\sigma^{r-1}(e_x^{r-1}), t)\bar{\Gamma}_0 \end{matrix} \right\| \tag{4.7}$$

is an $n \times n$ partitioned matrix function.

Lemma 1. Assume that the following conditions hold:
(1) the matrix $\bar{\Gamma}_0$ (4.3), (4.4) has $m \times m$ blocks:

$$\bar{\Gamma}_{0k} = \Gamma_{0k} - \Gamma_{0,k+1} S_{k+1,k} = O, \quad k = 1, \ldots, r-1; \quad \bar{\Gamma}_{0r} = \Gamma_{0r} \equiv S_{r+1,r} \tag{4.8}$$

where $S_{k+1,k}$ $(k = 1, \ldots, r)$ are $m \times m$ blocks, and the $m \times m$ blocks Γ_{0k} $(k = 1, \ldots, r)$ of the $m \times n$ matrix Γ_0 (1.17) may be represented in the form

$$\begin{aligned} &\Gamma_{0k} = \Gamma_{0,k+1} S_{k+1,k} = S_{r+1,r} S_{r,r-1} \ldots S_{k+1,k}, \quad k = 1, \ldots, r-1 \\ &\Gamma_{0r} \equiv S_{r+1,r} = \bar{\Gamma}_{0r} \end{aligned} \tag{4.9}$$

so that the matrix $\bar{\Gamma}_0$ (4.3), (4.4) has the form

$$\bar{\Gamma}_0 = \|O, \bar{\Gamma}_{0r}\| = \|O, S_{r+1,r}\| \tag{4.10}$$

(2) $S_{k+1,k}$ $(k = 1, \ldots, r)$ are non-singular constant $m \times m$ blocks representable in the form

$$S_{k+1,k} = B_k^{-1} \gamma_{S,k+1,k}, \quad k = 1, \ldots, r \tag{4.11}$$

where B_k ($k = 1, \ldots, r$) are the matrices defined by Eqs (1.10) and (1.13); $\gamma_{S,k+1,k}$ ($k = 1, \ldots, r$) are certain real numbers satisfying the inequalities

$$\begin{aligned}
&\gamma_{S,k+1,k} > 0, \quad k = 1, \ldots, r-1; \quad \gamma_{S,r+1,r} < 0 \\
&(|\gamma_{S,k+1,k}| = \gamma_{S,k+1,k} > 0,\ k = 1, \ldots, r-1; \quad |\gamma_{S,r+1,r}| = -\gamma_{S,r+1,r} > 0) \\
&\gamma_{S21} > \gamma_{0S21} = [2\underline{\lambda}(A_1)]^{-1}(r-1) \\
&|\gamma_{S,k+1,k}| > \gamma_{0S,k+1,k} = [2\underline{\lambda}(A_k)]^{-1}\left[\bar{\beta}_{Gkk} + (r-k) + \sum_{l=1}^{k-1} \alpha_{Gkl}\right], \quad k = 2, \ldots, r
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
&0 < \underline{\lambda}(A_1) = \min_i \inf_{t \ge t_0} \lambda_i(A_1(t)) \\
&0 < \underline{\lambda}(A_k) = \min_i \inf_{e_x^{k-1} \in R^{(k-1)m},\, t \ge t_0} \lambda_i(A_k(\sigma^{k-1}(e_x^{k-1}), t)); \quad i = 1, \ldots, m
\end{aligned} \tag{4.13}$$

$\sigma^k \equiv \sigma^k(e_x^k)$ is the vector-valued function (2.17); $\lambda_i(A_k)$ are the eigenvalues of the matrix-valued functions A_k ($k = 1, \ldots, r$) (1.11), (1.2), respectively

$$\lambda_i(A_1) \equiv \lambda_i(A_1(t)), \quad \lambda_i(A_k) \equiv \lambda_i(A_k(\sigma^{k-1}(e_x^{k-1}), t)); \quad k = 2, \ldots, r; \quad i = 1, \ldots, m$$

$\bar{\beta}_{Gkk}$, α_{Gkl} ($k = 2, \ldots, r$; $l = 1, \ldots, k-1$) are non-negative real numbers:

$$\begin{aligned}
&\bar{\beta}_{G22} = \sup_{t \ge t_0} |\bar{G}_{22}(t)|, \quad \bar{G}_{22}(e_x^0, t) \equiv \bar{G}_{22}(t) \\
&\bar{\beta}_{Gkk} = \sup_{e_x^{k-2} \in R^{(k-2)m},\, t \ge t_0} |\bar{G}_{kk}(e_x^{k-2}, t)|, \quad k = 3, \ldots, r \\
&\bar{G}_{kk}(e_x^{k-2}, t) = S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}), t) + \\
&+ [S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}), t)]^*, \quad \bar{G}_{k1}(e_x^0, t) \equiv \bar{G}_{k1}(t); \quad k = 2, \ldots, r \\
&\bar{P}_{012}(\sigma^0(e_x^0), t) \equiv \bar{P}_{012}(t) \\
&\alpha_{Gk1} = \sup_{t \ge t_0} |G_{k1}(t)|^2, \quad k = 2, \ldots, r \\
&\alpha_{Gkl} = \sup_{e_x^{l-1} \in R^{(l-1)m},\, t \ge t_0} |\bar{G}_{kl}(e_x^{l-1}, t)|, \quad k = 3, \ldots, r; \quad l = 2, \ldots, k-1
\end{aligned} \tag{4.14}$$

Then the equilibrium position $e_x = 0$ of the NCDS (4.1), (2.8), (2.15)–(2.19) closed by the control law e_u (4.2)–(4.14), (4.8)–(4.14) with linear feedback depending on the state e_x is stabilizable, so that the following propositions hold:

(1) the equilibrium position $e_u = 0$ of the transient equations (4.5)–(4.14), (1.10)–(1.13) (in the above closed-loop system) are globally asymptotically Lyapunov-stable;

(2) the solution $e_x(t)$ of this system satisfies the estimate

$$|e_x(t)| \le e^{-\alpha_0(t-t_0)}|e_x(t_0)|, \quad t \ge t_0 \tag{4.15}$$

where α_0 is the positive real number

$$\begin{aligned} &\alpha_0 = \min_k \alpha_{exk}, \quad k = 1, \dots, r; \quad \alpha_{ex1} = \frac{1}{2}[\alpha_{G11} - (r-1)] > 0 \\ &\alpha_{exk} = \frac{1}{2}\left[\alpha_{Gkk} - (r-k) - \sum_{l=1}^{k-1} \alpha_{Gkl}\right] > 0, \quad k = 2, \dots, r \\ &\alpha_{G11} = 2\gamma_{S21}\underline{\lambda}(A_1) > 0, \quad \alpha_{Gkk} = 2\underline{\lambda}(A_k)|\gamma_{S,k+1,k}| - \bar{\beta}_{Gkk} > 0, \quad k = 2, \dots, r \end{aligned} \tag{4.16}$$

Proof. We first note that in the transient equations (4.5)–(4.7), where P is the matrix-valued function (4.6) in which P_1 is given by (2.15)–(1.18), the matrix-valued function P_2 (4.7) (taking into account that the blocks $\bar{\Gamma}_{0k}$ and Γ_{0k} ($k = 1, \dots, r$), respectively, of the matric $\bar{\Gamma}_0$ (4.3), (4.4), (4.8), (4.10) and Γ_0 (1.17), (4.9) satisfy relations (4.8) and (4.9) [of the first condition of the lemma]) has the form

$$P_2(e_x^{r-1}, t) = \operatorname{diag}(O, P_{2rr}(e_x^{r-1}, t)), \quad P_{2rr}(e_x^{r-1}, t) = \bar{P}_{0r,r+1}(\sigma^{r-1}(e_x^{r-1}), t)\bar{\Gamma}_{0r} \tag{4.17}$$

We now consider the Lyapunov function

$$V(e_x) = |e_x|^2 = e_x^* e_x \tag{4.18}$$

and evaluate the derivative with respect to time t of the function $V(e_x(t))$ (4.18) by virtue of the transient equations (4.5)–(4.14), (1.10)–(1.13), along the non-trivial solution $e_x(t)$ of the system, taking into account that the blocks $S_{k+1,k}$ ($k = 1, \dots, r$) satisfy relations (4.11)–(4.14) (of the second condition of the lemma). We finally obtain

$$\dot{V}(e_x(t)) = W(e_x(t), t), \quad t \ge t_0 \tag{4.19}$$

where

$$W(e_x, t) = e_x^* G(e_x^{r-1}, t) e_x \tag{4.20}$$

is a quadratic form in which (taking relations (4.6), (2.15)–(2.18) and (4.17) into account)

$$G(e_x^{r-1}, t) = G^*(e_x^{r-1}, t) = P(e_x^{r-1}, t) + P^*(e_x^{r-1}, t) = \|G_{kl}\|_{k,l=1,\dots,r} \tag{4.21}$$

is a symmetric $n \times n$ matrix-valued function in which G_{kl} ($k, l = 1, \dots, r$) are $m \times m$ blocks:

$$\begin{aligned} &G_{11} \equiv G_{11}(t) = G_{11}^*(t) \equiv G_{11}(e_x^0, x) = P_{11}(t) + P_{11}^*(t) = -2\gamma_{S21}A_1(t) \\ &G_{kk} \equiv G_{kk}(e_x^{k-1}, t) = G_{kk}^*(e_x^{k-1}, t) = P_{kk}(e_x^{k-1}, t) + P_{kk}^*(e_x^{k-1}, t) = \\ &= -2A_k(\sigma^{k-1}(e_x^{k-1}), t)|\gamma_{S,k+1,k}| + \bar{G}_{kk}(e_x^{k-2}, t), \quad k = 2, \dots, r \\ &G_{1k}(t) = G_{k1}^*(t), \quad G_{k1}(t) = P_{1k1}(t), \quad k = 3, \dots, r; \quad G_{12}(t) = G_{21}^*(t) \\ &G_{21}(t) = P_{121}(t) + P_{112}^*(t), \quad G_{kl}(e_x^{l-1}, t) = P_{1kl}(e_x^{l-1}, t) \\ &G_{lk}(e_x^{l-1}, t) = G_{kl}^*(e_x^{l-1}, t); \quad k = 4, \dots, r; \quad l = 2, \dots, k-2 \\ &G_{k,k+1}(e_x^{k-1}, t) = P_{1,k,k+1}(e_x^{k-1}, t) + P_{1,k+1,k}^*(e_x^{k-1}, t) \\ &G_{k+1,k}(e_x^{k-1}, t) = G_{k,k+1}^*(e_x^{k-1}, t); \quad k = 2, \dots, r-1 \end{aligned} \tag{4.22}$$

where

$$P_{11}(t) = P_{111}(t) = -\bar{P}_{012}(t)S_{21} = -[A_1(t)B_1]S_{21} = -[A_1(t)B_1][B_1^{-1}\gamma_{S21}] = -A_1(t)\gamma_{S21}$$

$$
\begin{aligned}
&P_{kk}(e_x^{k-1},t) = P_{1kk}(e_x^{k-1},t) = \\
&= S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}),t) - \bar{P}_{0,k,k+1}(\sigma^{k-1}(e_x^{k-1}),t)S_{k+1,k} = \\
&= S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}),t) - [A_k(\sigma^{k-1}(e_x^{k-1}),t)B_k][B_k^{-1}\gamma_{S,k+1,k}] = \\
&= S_{k,k-1}\bar{P}_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}),t) - A_k(\sigma^{k-1}(e_x^{k-1}),t)|\gamma_{S,k+1,k}| \\
&|\gamma_{S,k+1,k}| = \gamma_{S,k+1,k} > 0; \quad k = 2,\dots,r-1 \qquad (4.23)\\
&P_{rr}(e_x^{r-1},t) = P_{1rr}(e_x^{r-2},t) + P_{2rr}(e_x^{r-1},t) = \\
&= S_{r,r-1}\bar{P}_{0,r-1,r}(\sigma^{r-2}(e_x^{r-2}),t) + \bar{P}_{0,r,r+1}(\sigma^{r-1}(e_x^{r-1}),t)S_{r+1,r} = \\
&= S_{r,r-1}\bar{P}_{0,r-1,r}(\sigma^{r-2}(e_x^{r-2}),t) + [A_k(\sigma^{r-1}(e_x^{r-1}),t)B_r][B_r^{-1}\gamma_{S,r+1,r}] = \\
&= S_{r,r-1}\bar{P}_{0,r-1,r}(\sigma^{r-2}(e_x^{r-2}),t) - A_r(\sigma^{r-1}(e_r^{r-1}),t)|\gamma_{S,r+1,r}| \\
&|\gamma_{S,r+1,r}| = -\gamma_{S,r+1,r} > 0, \quad \gamma_{S,r+1,r} < 0
\end{aligned}
$$

$\sigma^k \equiv \sigma^k(e_x^k)$ is the mk-vector-valued function (2.17), the $m \times m$ blocks $\bar{G}_{kk}$ ($k = 2, \dots, r$) are defined by the formulae in (4.14), and by expressions (4.8) and (4.9)

$$\bar{\Gamma}_{0r} = \Gamma_{0r} = S_{r+1,r}$$

We will estimate the quadratic form $W(e_x, t)$ (4.20)–(4.23).

To that end, we first observe that, since the matrix-valued functions A_k ($k = 1, \dots, r$) are positive-definite and estimates (1.11) are valid, we have the estimates

$$
\begin{aligned}
&\underline{\lambda}(A_1)|e_{x1}|^2 \le e_{x1}^* A_1(t)e_{x1} \le \bar{\lambda}(A_1)|e_{x1}|^2, \quad \forall t \ge t_0 \\
&\underline{\lambda}(A_k)|e_{xk}|^2 \le e_{xk}^* A_k(\sigma^{k-1}(e_x^{k-1}),t)e_{xk} \le \bar{\lambda}(A_k)|e_{xk}|^2 \\
&\forall e_x^{k-1} \in R^{(k-1)m}, \quad \forall t \ge t_0, \quad k = 2,\dots,r
\end{aligned} \tag{4.24}
$$

where $\underline{\lambda}(A_k) > 0$ ($k = 1, \dots, r$) are the real numbers (4.13) and

$$
\begin{aligned}
&0 < \bar{\lambda}(A_1) = \max_i \sup_{t \ge t_0} \lambda_i(A_1(t)) \\
&0 < \bar{\lambda}(A_k) = \max_i \sup_{e_x^{k-1} \in R^{(k-1)m},\, t \ge t_0} \lambda_i(A_k(\sigma^{k-1}(e_x^{k-1}),t)) \\
&i = 1,\dots,m; \quad k = 2,\dots,r
\end{aligned} \tag{4.25}
$$

We then estimate the quadratic forms

$$W_{11}(e_{x1},t) = e_{x1}^* G_{11}(t)e_{x1}, \quad W_{kk}(e_x^k,t) = e_{xk}^* G_{kk}(e_x^{k-1},t)e_{xk}, \quad k = 2,\dots,r \tag{4.26}$$

Taking relations (4.22) and (4.14) into account, we obtain

$$
\begin{aligned}
&W_{11}(e_{x1},t) = e_{x1}^* G_{11}(t)e_{x1} = -2e_{x1}^*\gamma_{S21}A_1(t)e_{x1} \le -\alpha_{G11}|e_{x1}|^2 \\
&W_{kk}(e_x^k,t) = e_{xk}^* G_{kk}(e_x^{k-1},t)e_{xk} = \\
&= e_{xk}^*[-2A_k(\sigma^{k-1}(e_x^{k-1}),t)|\gamma_{S,k+1,k}| + \bar{G}_{kk}(e_x^{k-2},t)]e_{xk} \le \\
&\le -[2\underline{\lambda}(A_k)|\gamma_{S,k+1,k}| - |\bar{G}_{kk}(e_x^{k-2},t)|]|e_{xk}|^2 \le -\alpha_{Gkk}|e_{xk}|^2, \quad k = 2,\dots,r
\end{aligned} \tag{4.27}
$$

where α_{Gkk} ($k = 1, \dots, r$) are the positive real numbers (4.16), (4.12)–(4.14).

Now, taking relations (4.22), (4.23), (4.26) and (4.27) into account and using the inequalities

$$2e_{xk}^{*}G_{kl}(e_x^{l-1}, t)e_{xl} \le 2[|e_{xk}^{*}||G_{kl}(e_x^{l-1}, t)|]|e_{xl}| \le |e_{xl}|^2 + \alpha_{Gkl}|e_{xk}|^2$$

$$k = 2, \ldots, r; \quad l = 1, \ldots, k-1$$

where α_{Gkl} are the non-negative real numbers defined by the last two formulae of (4.14), we estimate the quadratic form $W(e_x, t)$ (4.20)–(3.24)

$$W(e_x, t) = e_x^{*}G(e_x^{r-1}, t)e_x = \sum_{k=1}^{r} e_{xk}^{*}G_{kk}(e_x^{k-1}, t)e_{xk} + 2\sum_{k=2}^{r}\left[\sum_{l=1}^{k-1} e_{xk}^{*}G_{kl}(e_x^{l-1}, t)e_{xl}\right] \le$$

$$\le \sum_{k=1}^{r} e_{xk}^{*}G_{kk}(e_x^{k-1}, t)e_{xk} + \sum_{k=2}^{r}\left\{\sum_{l=1}^{k-1}[|e_{xl}|^2 + \alpha_{Gkl}|e_{kl}|^2]\right\} =$$

$$= e_{x1}^{*}[G_{11}(t) + (r-1)I_m]e_{x1} + \sum_{k=2}^{r} e_{xk}^{*}\left\{G_{kk}(e_x^{k-1}, t) + \left[(r-k) + \sum_{l=1}^{k-1}\alpha_{Gkl}\right]I_m\right\}e_{xk} \le$$

$$\le [-\alpha_{G11} + (r-1)]|e_{x1}|^2 + \sum_{k=2}^{r}\left[-\alpha_{Gkk} + (r-k) + \sum_{l=1}^{k-1}\alpha_{Gkl}\right]|e_{xk}|^2 =$$

$$= -2\sum_{k=1}^{r}\alpha_{exk}|e_{xk}|^2 \le -2\alpha_0|e_x|^2 = -2\alpha_0 V(e_x(t)), \quad t \ge t_0 \tag{4.28}$$

where $\alpha_{exk} > 0$ $(k = 1, \ldots, r)$, $\alpha_0 > 0$ are the real numbers from (4.16), (4.11)–(4.14). It follows from relations (4.19) and (4.28) that

$$\dot{V}(e_x(t)) = W(e_x(t), t) \le -2\alpha_0 V(e_x(t)), \quad t \ge t_0 \tag{4.29}$$

from which we find

$$V(e_x(t)) \le V(e_x(t_0))\exp[-2\alpha_0(t - t_0)], \quad t \ge t_0$$

Hence, again using relations (4.18), we obtain

$$|e_x(t)|^2 \le |e_x(t_0)|^2\exp[-2\alpha_0(t - t_0)], \quad t \ge t_0$$

Consequently, the equilibrium position $e_x = 0$ of the transient equations – system (4.5)–(4.14), (1.10)–(1.13) – is globally asymptotically Lyapunov-stable, and the solution $e_x(t)$ satisfies the estimate

$$|e_x(t)| \le \exp[-\alpha_0(t - t_0)]|e_x(t_0)|, \quad t \ge t_0 \tag{4.30}$$

that is, the equilibrium position $e_x = 0$ of the NCDS (4.1), (2.15)–(2.19), closed by the control law e_u (4.2)–(4.4), (4.8)–(4.14), with linear feedback depending on the state e_x, is stabilizable. This completes the proof of Lemma 1.

2. Let us consider the behaviour of the solution $e_x(t)$ of the NCDS of special form (2.13)–(2.22)

$$\dot{e}_x = P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0$$

(where e_x is the state vector (2.8) of the system, P_1 and Q_1 are the matrix-valued functions (2.15)–(2.18) and (2.19), and g_{ex} is the vector-valued function (2.2)–(2.22)), closed by a control law e_u (1.18), (1.17) represented, if relations (2.8)–(2.12) are taken into account, in the form (4.2)–(4.4)

$$e_u = \Gamma_0 e = e_{ux} \equiv \Gamma_0 R e_x = \bar{\Gamma}_0 e_x \quad (\bar{\Gamma}_0 = \Gamma_0 R)$$

and transient equations (in the above closed-loop system) of the form

$$\dot{e}_x = P(e_x^{r-1}, t)e_x + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.31}$$

where P is then $n \times n$ matrix-valued function (4.6), (2.15)–(2.18), (4.7) and g_{ex} is the vector-valued function (2.20)–(2.22).

Lemma 2. Assume that the conditions of Lemma 1 are satisfied and that in estimate (2.21) for the vector-valued function g_{ex} (2.20) the coefficients k_{gexj} ($j = 1, 2$) (2.22) are such that

$$\begin{aligned} 0 \le k_{gex1} &= |S||R|k_{ge1} < (1-\nu_0)\alpha_0, \quad 0 < \nu_0 < 1 \\ 0 < k_{gex2} &= |S||R|^2 k_{ge2} < \infty \end{aligned} \tag{4.32}$$

where k_{ge1}, k_{ge2} and α_0 are the constants defined by (2.7), (1.14), (1.15) and (4.16), (4.11)–(4.14).

Then the NCDS of special form (2.13)–(2.22), (4.32), closed by a control law e_u (4.2)–(4.4), (4.8)–(4.14), with linear feedback depending on the state e_x, is stabilizable, so that the following propositions hold for the solution $e_x(t)$ of the transient equation (in the above closed-loop NCDS) – the system (4.31), (4.6), (2.15)–(2.18), (4.7), (2.20)–(2.22), (4.32):

(1) the equilibrium position $e_x = 0$ of system (4.31), (4.6), (2.15)–(2.18), (4.7), (2.20)–(2.22), (4.32) is asymptotically stable in the large;

(2) the DAS of the equilibrium position $e_x = 0$ of system (4.31), (4.6), (2.15)–(21.8), (4.7), (2.20)–(2.22), (4.32) is the set

$$\Omega_{ex0} = \{e_x \in R^n: \upsilon(e_x) = |e_x| < \rho_{ex0}\} \tag{4.33}$$

where

$$\rho_{ex0} = k_{gex2}^{-1}[(1-\nu_0)\alpha_0 - k_{gex1}] > 0, \quad 0 < \nu_0 < 1 \tag{4.34}$$

σ_0 being the constant defined by relations (4.16), (4.11)–(4.14);

(3) a non-trivial solution $e_x(t)$ of system (4.31), (4.6), (2.15)–(2.18), (4.7), (2.20)–(2.22), (4.32) satisfies the limit

$$|e_x(t)| \le e^{-\gamma_0(t-t_0)}|e_x(t_0)|, \quad e_x(t_0) \in \Omega_{ex0}, \quad t \ge t_0; \quad \gamma_0 = \nu_0\alpha_0 \tag{4.35}$$

where Ω_{ex0} is the set (4.33), (4.34).

Proof. We shall show that the assumptions of the Auxiliary Lemma hold for system (4.31), (4.6), (2.15)–(2.18), (4.7), (2.20)–(2.22), (4.32), written in the form

$$\dot{e}_x = f(e_x, t) + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.36}$$

where

$$f(e_x, t) \equiv P(e_x^{r-1}, t)e_x \tag{4.37}$$

and g_{ex} is the vector-valued function (2.20)–(2.22), (4.32).

Consider the Lyapunov function

$$\upsilon(e_x) = |e_x| = (V(e_x))^{1/2} = (e_x^* e_x)^{1/2} \tag{4.38}$$

where $V(e_x)$ is the function (4.18).

Conditions 1 and 2 of the Auxiliary Lemma hold for the function $\upsilon(e_x)$ (4.38), (4.18), where

$$\varepsilon_{\upsilon i} = 1, \quad i = 1, 2, 3 \tag{4.39}$$

Because of relations (4.39) and (4.32), the coefficients k_{gexj} ($j = 1, 2$) (2.22) in the estimate (2.21) for the vector-valued function g_{ex} (2.20) satisfy the estimate in condition 3 of the Auxiliary Lemma, where α_0 is the constant defined by relations (4.16), (4.11)–(4.14),

$$k_{g1} = k_{gex1} = |S||R|k_{ge1} \ge 0, \quad k_{g2} = k_{gex2} = |S||R|^2 k_{ge2} > 0, \quad \varepsilon_{\upsilon i} = 1, \quad i = 1, 2, 3$$

where $k_{ge1} \ge 0$, $k_{ge2} > 0$ are the constants defined by relations (2.7), (1.14) and (1.15).

Since the conditions of Lemma 1 hold, it follows that the derivative with respect to t of the function $V(e_x(t))$ (4.18) along trajectories of system (4.5)–(4.7), (2.15)–(2.18), written in the form

$$\dot{e}_x = f(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{4.40}$$

where f is the vector-valued function (4.37), satisfy relations (4.19), (4.20), (4.28) and (4.29)

$$\dot{V}(e_x(t)) = W(e_x(t)) \leq -2\alpha_0 V(e_x(t)), \quad t \geq t_0 \tag{4.41}$$

where $W(e_x(t))$ is the function (4.20)–(4.23) and α_0 is the positive number defined in relations (4.16), (4.11)–(4.14).

Taking the estimate (4.41) into account, let us evaluate the derivative with respect to t of the function $\upsilon(e_x(t))$ (4.38), (4.18) along trajectories of system (4.40), (4.37). We obtain

$$\dot{\upsilon}(e_x(t)) = [(V(e_x(t)))^{1/2}]^{\cdot} = \frac{\dot{V}(e_x(t))}{2(V(e_x(t)))^{1/2}} = \frac{1}{2\upsilon(e_x(t))}\dot{V}(e_x(t)) =$$

$$= \frac{1}{2\upsilon(e_x(t))}\frac{\partial V(e_x)}{\partial e_x} f(e_x, t) \leq -\frac{1}{2\upsilon(e_x(t))} 2\alpha_0 V(e_x(t)) =$$

$$= -\frac{1}{2\upsilon(e_x(t))} 2\alpha_0 [\upsilon(e_x(t))]^2 = -\alpha_0 \upsilon(e_x(t)), \quad t \geq t_0 \tag{4.42}$$

and hence the fourth condition of the Auxiliary Lemma is holds.

Thus, system (4.36), (4.37), (2.20)–(2.22), (4.23) satisfies all the conditions of Auxiliary Lemma and consequently the conclusions of the lemma are true; but by (4.38) and (4.18), those conclusions are identical with the assertions of Lemma 2 for system (4.31), (4.6), (2.15)–(2.18), (4.7), (2.20)–(2.22), (4.32), written as the system (4.36), (4.37), (2.20)–(2.22), (4.32), Lemma 2 is proved.

3. We now consider the behaviour of a solution $e(t)$ of a NCDS in canonical form

$$\dot{e} = \bar{P}_0(e^{r-2}, t)e + \bar{Q}_0(e^{r-1}, t)e_u, \quad e(t_0) = e_0, \quad t \geq t_0 \tag{4.43}$$

(where $\bar{P}_0$ and $\bar{Q}_0$ are the matrix-valued functions (2.3) and (2.4)), closed by the control law e_u (1.18), (1.17)

$$e_u = \Gamma_0 e$$

with linear feedback depending on the state e, and transient equations (in the above closed-loop system)

$$\dot{e} = P_0(e^{r-1}, t)e, \quad e(t_0) = e_0, \quad t \geq t_0 \tag{4.44}$$

where $P_0(e^{r-1}, t)$ is the $n \times n$ matrix-valued function

$$P_0(e^{r-1}, t) = \bar{P}_0(e^{r-2}, t) + \bar{Q}_0(e^{r-1}, t)\Gamma_0 = P_{01}(e^{r-2}, t) + P_{02}(e^{r-1}, t) \tag{4.45}$$

The $n \times n$ partitioned matrix-valued functions have the form

$$P_{01}(e^{r-2}, t) = \bar{P}_0(e^{r-2}, t) \tag{4.46}$$

$$P_{02}(e^{r-1}, t) = \bar{Q}_0(e^{r-1}, t)\Gamma_0 = \left\| \begin{matrix} O \\ \bar{P}_{0r, r+1}(e^{r-1}, t)\Gamma_0 \end{matrix} \right\| \tag{4.47}$$

Theorem 1. Assume that the conditions of Lemma 1 hold.

Then an NCDS of canonical form (4.43), (2.3), (2.4), closed by the control law e_u (1.18), (1.17), (4.8)–(4.14), with linear feedback depending on the state e, is stabilizable, so that a solution $e(t)$ of the transient equations (in the above closed-loop NCDS) – the system (4.44)–(4.47), (1.17), (4.9)–(4.4) – satisfies the following assertions:

(1) the equilibrium position $e = 0$ of system (4.44)–(4.47), (1.17), (4.9)–(4.14) is globally asymptotically stable;

(2) a non-trivial solution $e(t)$ of system (4.44)–(4.47), (1.17), (4.9)–(4.14) satisfies the estimate

$$|e(t)| \le \beta_0 e^{-\alpha_0(t-t_0)}|e(t_0)|, \quad e(t_0) = e_0, \quad t \ge t_0; \quad \beta_0 = |R||S| \tag{4.48}$$

where α_0 is the positive number defined in relations (4.17), (4.11)–(4.14).

Proof. We first apply a non-singular linear transformation of coordinates of the state space of the form (2.8)–(2.12)

$$e_x = Se \quad (e = S^{-1}e_x = Re_x)$$

to the NCDS in canonical form (4.43), (2.3), (2.4), bringing it to the form of the NCDS (4.1), (2.15)–(2.19)

$$\dot{e}_x = P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u, \quad e_x(t_0) = e_{x0}, \quad t \ge t_0$$

The first and second conditions of Lemma 1 hold for the NCDS (4.1), (2.15)–(2.19), so that the conclusions of that lemma are also true for the system.

It follows from the assertions of Lemma 1, the non-singularity of the transformation of variables (2.8)–(2.12) and the estimates

$$|e| = |Re_x| \le |R||e_x|, \quad |e_x| = |Se| \le |S||e| \tag{4.49}$$

that the assertions of Theorem 1 hold for a NCDS (4.43), (2.3), (2.4) of canonical form, closed by a control law e_u (1.18), (1.17), (4.8)–(4.14) with linear feedback depending on the state e, and also for a solution $e(t)$ of the transient equation (in the aforementioned closed-loop system). This completes the proof of Theorem 1.

4. Finally, let us consider the behaviour of the solution $e(t)$ of the initial NCDS in deviations (2.1)–(2.7):

$$\dot{e} = \bar{P}_0(e^{r-2}, t)e + \bar{Q}_0(e^{r-1}, t)e_u + g_e(e, t), \quad e(t_0) = e_0, \quad t \ge t_0$$

closed by the control law e_u (1.18), (1.17)

$$e_u = \Gamma_0 e$$

with linear feedback with respect to the state e, and transient equation (in the aforementioned closed-loop system)

$$\dot{e} = P_0(e^{r-1}, t)e + g_e(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{4.50}$$

where P_0 is the matrix-valued function (4.45)–(4.47) and g_e is the vector-valued function (3.5)–(2.7).

Theorem 2. Assume that the conditions of Lemma 2 are satisfied, the vector-valued function g_e (2.5) satisfies the estimate (2.6), (2.7), and in the estimate (2.21) for the vector-valued function g_{ex} (2.20) the coefficients k_{gexj} ($j = 1, 2$) (2.22) satisfy inequalities (4.32).

Then the initial NCDS is deviations (2.1)–(2.7), closed by a control law e_u (1.18), (1.17), (4.8)–(4.14) with linear feedback depending on the state e, is stabilizable, so that the following propositions hold for a solution $e(t)$ of the transient equation (in the aforementioned closed-loop NCDS) – system (4.50), (4.45)–(4.47), (1.17), (4.8)–(4.14), (2.5)–(2.7):

(1) the equilibrium position $e = 0$ of system (4.50), (4.45)–(4.47), (1.17), (4.8)–(4.14), (2.5)–(2.7) is a symptotically stable in the large;

(2) the DAS of the equilibrium position $e = 0$ of system (4.50), (4.45)–(4.47), (1.17), (4.8)–(4.14), (2.5)–(2.7) is the set

$$\Omega_{e0} = \{e \in R^n: e = Re_x, e_x \in \Omega_{ex0}\} \tag{4.51}$$

where Ω_{ex0} is the set (4.33), (4.34);

(3) the following estimate holds for a non-trivial solution $e(t)$ of system (4.50), (4.45)–(4.47), (1.17), (4.8)–(4.14), (2.5)–(2.7).

$$|e(t)| \le \beta_0 e^{-\gamma_0(t-t_0)}|e(t_0)|, \quad e(t_0) \in \Omega_{e0}, \quad t \ge t_0 \tag{4.52}$$

where

$$\beta_0 = |R||S|, \quad \gamma_0 = \nu_0\alpha_0, \quad 0 < \nu_0 < 1$$

α_0 is the positive number defined in relations (4.16), (4.11)–(4.14), and Ω_{e0} is the set (4.51).

Proof. We first apply a non-singular linear transformation of coordinates in the state space of the type (2.8)–(2.12),

$$e_x = Se \quad (e = S^{-1}e_x = Re_x)$$

to the initial NCDS is deviations (2.1)–(2.7), to obtain a NCDS (2.13)–(2.22) of special form

$$\dot{e}_x = P_1(e_x^{r-2}, t)e_x + Q_1(e_x^{r-1}, t)e_u + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0$$

In estimate (2.21) for the vector-valued function g_{ex} (2.20), the coefficients k_{gexj} $(j = 1, 2)$ (2.22) satisfy inequalities (4.32).

Since the NCDS of special form (2.13)–(2.22), (4.32) satisfies the conditions of Lemma 2, the conclusions of that lemma hold for this system.

It follows from the assertions of Lemma 1, the non-singularity of the transformation of variables (2.8)–(2.12) and the estimates (4.49) and (4.32) that the conclusions (analogous to those of Lemma 2) formulated in Theorem 2 hold for the initial NCDS in deviations (2.1)–(2.7), closed by a control law e_u (1.18), (1.17), (4.8)–(4.14) with linear feedback depending on the state e, and also for a solution $e(t)$ of the transient equation (in the aforementioned closed-loop NCDS) – the system (4.50), (4.45)–(4.47), (1.17), (4.8)–(4.14), (2.5)–(2.7). This completes the proof of Theorem 2.

5. APPENDIX

The equations of dynamics for an NCDS of the type of an electromechanical system (such as an electromechanical manipulator robot), comprising a slave mechanism (SM), electric drives (ED) based on DC motors, and with rigid reductors, have the following form [7]

$$\begin{gathered}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial \Pi}{\partial q}\right]^* + Q_c \equiv A_0(q)\ddot{q} + b_0(q, \dot{q}, t) = Q_u \\ J\ddot{\alpha} + k_0\dot{\alpha} + i_p^{-1}\eta_p^{-1}Q_u = k_M I_a, \quad L\dot{I}_a + RI_a + k_e\dot{\alpha} = u\end{gathered} \tag{5.1}$$

The first equation describes the dynamics of the SM in the form of Lagrange equations of the second kind and the second and third equations describe the dynamics of the ED. Here $q = \mathrm{col}(q_1, \dots, q_m)$ is the m-dimensional vector of generalized coordinates $q_1, \dots, q_m$ of the SM, m is the number of degrees of freedom (mobility) of the SM and $A_0(q)$ is the continuously differentiable symmetric positive-definite $m \times m$ matrix-valued function of kinetic energy $T = \dot{q}^* A_0(q)\dot{q}/2$ of the SM, where

$$|A_0(q)| \le k_{A0}, \quad \forall q \in R^m \tag{5.2}$$

for some constant $0 < k_{A0} < \infty$; analogous estimates hold for the partial derivatives of its elements; $a_{0ij}(q)$ $(i, j = 1, \dots, m)$ are scalar functions of their arguments;

$$b_0(q, \dot{q}, t) = \dot{A}_0(q)\dot{q} - \frac{1}{2}\left[\frac{\partial(\dot{q}^* A_0(q)\dot{q})}{\partial q}\right]^* + Q_\Pi + Q_c \tag{5.3}$$

$$Q_\Pi = Q_\Pi(q) = \left[\frac{\partial\Pi(q)}{\partial q}\right]^* = \mathrm{col}(Q_{\Pi 1}(q_1), \dots, Q_{\Pi m}(q_m))$$
$$Q_{\Pi i}(q_i) = \frac{\partial\Pi(q)}{\partial q_i}, \quad i = 1, \dots, m \tag{5.4}$$

$$|Q_\Pi(q)| \le k_{Q\Pi}, \quad \forall q \in R^m \tag{5.5}$$

$$Q_c = Q_c(q, \dot q, t) = \Theta_c(q, t)\dot q \tag{5.6}$$

Q_Π is the m-dimensional vector of potential forces driving the SM, $\Pi = \Pi(q)$ is the potential energy of the SM, $Q_{\Pi i}(q_i)$ are continuously differentiable functions, $k_{Q\Pi} \ge 0$ is a constant, Q_c is the m-dimensional vector of generalized forces (torques) of resistance acting on the degree of mobility of the SM, $\bar\Theta_c(q, t) = (\Theta_c(q, t) + \Theta_c^*(q, t))/2$ is a continuously differentiable, symmetric, positive-definite $m \times m$ matrix-valued function, I_a is the m-dimensional vector of currents in the armature circuits of the DC motors, $u = \mathrm{col}(u_1, \dots, u_m)$ is the m-dimensional vector of controls – controlling voltages applied to the armature circuits of the DC motors, $Q_u = \mathrm{col}(Q_{u1}, \dots, Q_{um})$ is the m-dimensional vector of generalized forces (torques) applied to the degrees of mobility of the SM, J, k_0, k_M, L, R, k_e are the diagonal matrices of electromechanical parameters of the DC motors, which are positive real quantities, i_p and η_p are the diagonal matrices of transfer coefficients and coefficients of useful action of the reductors and $\alpha = i_p q$, where α is the m-vector of angles of rotation of the shafts of the motors.

The equations of motion of an NCDS of type (5.1)–(5.6), written in terms of deviations e and e_u (1.4), are

$$e = z - z_p = \mathrm{col}(e_1, e_2, e_3), \quad z = \mathrm{col}(q, \dot q, I_a), \quad z_p = \mathrm{col}(q_p, \dot q_p, I_{ap})$$
$$(e_1 = q - q_p, e_2 = \dot q - \dot q_p, e_3 = I_a - I_{ap}), \quad z_p \in \Omega_{zp}$$
$$\Omega_{zp} = \{z_p = \mathrm{col}(q_p, \dot q_p, I_{ap}) \in R^{3m}: q_p \in R^m, \dot q_p \in R^m, I_{ap} \in R^m;$$
$$|\dot q_p(t)| \le k_{z2p} < \infty, \quad |I_{ap}(t)| \le k_{z3p} < \infty; \quad t \ge t_0\}; \ e_u = u - u_p \tag{5.7}$$

The deviations are measured from their programmed values z_p and u_p (where $0 < k_{z2p} < \infty$, $0 \le k_{z3p} < \infty$ are constants), represented in the form of system (1.5)–(1.10) with $n = 3m$, $r = 3$ and

$$F_{e1}(e^2, t) = \dot q - \dot q_p = g_{e1}(e^1, t) + \bar P_{012}(t)e_2$$

$$(g_{e1}(e^1, t) = 0, \bar P_{012}(t) = A_1(t)B_1 = I_m, A_1(t) = B_1 = I_m)$$

$$F_{e2}(e^3, t) = A^{-1}(q)(k_M I_a - b(q, \dot q, t)) - A^{-1}(q_p)(k_M I_{ap} - b(q_p, \dot q_p, t)) =$$

$$= g_{e2}(e^2, t) + \bar P_{023}(e^1, t)e_3$$

$$g_{e2}(e^2, t) = A^{-1}(q)(k_M I_{ap} - b(q, \dot q, t)) - A^{-1}(q_p)(k_M I_{ap} - b(q_p, \dot q_p, t)) =$$

$$= [A^{-1}(q) - A^{-1}(q_p)](-b(q_p, \dot q_p, t) + k_M I_{ap}) - A^{-1}(q)\Delta b(e_1, e_2, t) =$$

$$= \Delta\bar A(e_1, t)(-b(q_p, \dot q_p, t) + k_M I_{ap}) - A^{-1}(q)\Delta b(e_1, e_2, t)$$

$$\Delta\bar A(e_1, t) = A^{-1}(q) - A^{-1}(q_p) = A^{-1}(e_1 + q_p) - A^{-1}(q_p), \quad q = e_1 + q_p \tag{5.8}$$

$$\bar P_{023}(e^2, t) = A_2(e^1, t)B_2 = (J i_p^2 \eta_p + A_0(q))^{-1}\eta_p i_p k_M = A^{-1}(q)k_M$$

$$A_2(e^1, t) = (J i_p^2 \eta_p + A_0(q))^{-1}, \quad B_2 = \eta_p i_p k_M$$

$$F_3(e, e_u, t) = L^{-1}(e_u - R(I_a - I_{ap}) - k_e i_p(\dot{q} - \dot{q}_p)) = g_{e3}(e^3, t) + \bar{P}_{034}(e^3, t)e_u$$

$$g_{e3}(e^3; t) = -L^{-1}(R(I_a - I_{ap}) + k_e i_p(\dot{q} - \dot{q}_p))$$

$$\bar{P}_{034}(e^2, t) = A_3(e^2, t)B_3 = L^{-1}, \quad A_3(e^2, t) = L^{-1}, \quad B_3 = I_m$$

$$A(q) = Ji_p + i_p^{-1}\eta_p^{-1}A_0(q) = i_p^{-1}\eta_p^{-1}(Ji_p^2\eta_p + A_0(q))$$

$$b(q, \dot{q}, t) = k_0 i_p \dot{q} + i_p^{-1}\eta_p^{-1}b_0(q, \dot{q}, t)$$

Here

$$|\Delta\bar{A}(e_1, t)| \le \bar{k}_{\Delta A}|e_1|, \quad \forall e_1 \in R^m, \quad t \ge t_0 \tag{5.9}$$

where $0 \le \bar{k}_{\Delta A} < \infty$ is a constant; analogous estimates hold for the partial derivatives of its elements: $\Delta\bar{a}_{ij}$ (e_1, t) $(i, j = 1, \dots, m)$ are scalar functions of their arguments, $A_2(e^1, t)$ is a symmetric, positive-definite matrix-valued function:

$$\begin{aligned} \Delta Q_\Pi(e_1, t) &= Q_\Pi(e_1 + q_p) - Q_\Pi(q_p) \\ |\Delta Q_\Pi(e_1, t)| &\le k_{\Delta Q\Pi}|e_1|, \quad \forall e_1 \in R^m, \quad t \ge t_0 \end{aligned} \tag{5.10}$$

$k_{\Delta Q\Pi} \ge 0$ is a certain constant;

$$\begin{aligned} \Delta Q_c(e_1, e_2, t) &= Q_c(q, \dot{q}, t) - Q_c(q_p, \dot{q}_p, t) = \\ &= \Theta_c(q, t)\dot{q} - \Theta_c(q_p, t)\dot{q}_p = \Theta_c(e_1 + q_p, t)e_2 + \Delta\Theta_c(e_1, t)\dot{q}_p \\ \Delta\Theta_c(e_1, t) &= \Theta_c(e_1 + q_p, t) - \Theta_c(q_p, t) \\ |\Delta Q_c(e_1, e_2, t)| &\le k_{\Delta Qc1}|e_1| + k_{\Delta Qc2}|e_2|, \quad \forall e_1, e_2 \in R^m, \quad t \ge t_0 \end{aligned} \tag{5.11}$$

$k_{\Delta Qc1} \ge 0$, $k_{\Delta Qc2} > 0$ are certain constants, and

$$\begin{aligned} \Delta b(e_1, e_2, t) &= b(q, \dot{q}, t) - b(q_p, \dot{q}_p, t) = k_0 i_p(\dot{q} - \dot{q}_p) + \\ &+ i_p^{-1}\eta_p^{-1}(b_0(q, \dot{q}, t) - b_0(q_p, \dot{q}_p, t)) = k_0 i_p e_2 + i_p^{-1}\eta_p^{-1}\Delta b_0(e_1, e_2, t) \end{aligned} \tag{5.12}$$

$$\Delta b_0(e_1, e_2, t) = b_0(e_1 + q_p, e_2 + \dot{q}_p, t) - b_0(q_p, \dot{q}_p, t) \tag{5.13}$$

Taking relations (5.1)–(5.3) into account, we obtain the following estimates

$$\begin{aligned} &|\Delta b(e_1, e_2, t)| \le k_{\Delta b1}|e_1| + k_{\Delta b2}|e_2| + k_{\Delta b22}|e_2|^2 \\ &|\Delta b_0(e_1, e_2, t)| \le k_{\Delta b01}|e_1| + k_{\Delta b02}|e_2| + k_{\Delta b022}|e_2|^2 \\ &\forall e_1, e_2 \in R^m, \quad t \ge t_0 \end{aligned} \tag{5.14}$$

where $k_{\Delta bj} \ge 0$, $k_{\Delta b22} > 0$, $k_{\Delta b0j} \ge 0$, $k_{\Delta b022} > 0$ are constants.

It follows from relations (5.1)–(5.14) that relations (1.11)–(1.15) hold, and consequently the NCDS (1.5)–(5.14) is a NCDS of the form (1.5)–(1.14).

Example. Consider an electromechanical robot whose *SM* is a three-dimensional two-stage manipulator [7] whose kinematic diagram is shown in Fig. 1. Here q_1 and q_2, the generalized coordinates of the SM, are the angles formed by the corresponding limbs – the degrees of mobility of the SM – with the axes of a fixed Cartesian system of coordinate *Oxyz*, l_i and m_i are the length and mass of the ith limb; r_1 is the radius of the shaft-hinge of the first limb – a homogeneous cylinder; r_2 is the distance of

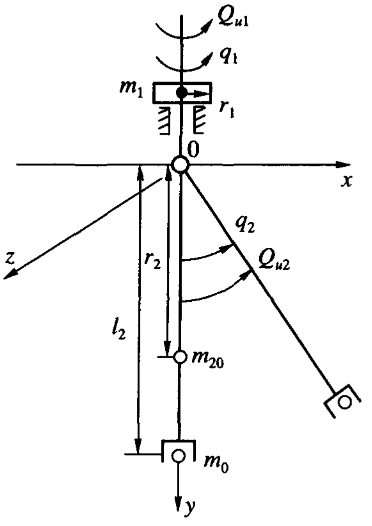

Fig. 1

the centre of gravity of the second limb (allowing for the mass m_0 of the load in its gripping device) from its axis of rotation; Q_{ui} is the torque of the load of the ith limb of the SM, $i = 1, 2$ and $m = 2$ is the number of degrees of freedom (mobility) of the SM.

In the equations of the dynamics of the SM of such a robot (see the first equation of system (5.1) and relations (5.3), (5.4) and (5.6)),

$$A_0(q) = \operatorname{diag}(2J_{01} + m_{20}r_2^2\sin^2 q_2, m_{20}r_2^2) \tag{5.15}$$

is the 2×2 diagonal matrix of the kinetic energy of the SM, where $J_{01} = m_1r_1^2/2$ is the moment of inertia of the first limb of the SM about its longitudinal axis of rotation, $m_{20} = m_2 + m_0$,

$$\Pi(q_2) = m_{20}\tilde{g}r_2(1 - \cos q_2) \tag{5.16}$$

is the potential energy of the SM, where $\tilde{g}$ is the gravitational acceleration, and

$$\Theta_c = \operatorname{diag}(k_{BT1}, k_{BT2}) \tag{5.17}$$

is a 2×2 diagonal matrix, where $k_{BTi} > 0$ ($i = 1, 2$) are the damping factors (viscous friction).

It can be proved that the estimates (5.2), (5.5), (5.9)–(5.14) hold for a robot of this description, whose equations of dynamics have the form (5.1)–(5.3), (5.7), (5.8), (5.15)–(5.17). It follows that relations (1.10)–(1.15) hold, and hence that the NCDS (5.1)–(5.3), (5.7), (5.8), (5.15)–(5.17) is of the form (1.1), (1.5)–(1.15).

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